PMM U.S.S.R.,Vol.47, No.1,pp.110-115,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.3

PROBLEM ON THE CONTACT OF TWO ELASTIC PLATES*

A.M. KHLUDNEV

The problem is considered of the contact between two elastic plates without a priori assumptions about the form of the contact set. The geometry of the problem results in a natural definition of a convex set of allowable displacements. The solution yields the minimum of the functional energy in this set and satisfies a variational inequality. The main result is the proof of the connectedness of the non-contact domain for an appropriate condition on the external load. A condition for no internal points of the contact set is also found. Analogous results hold for the contact between a shallow shell and a plate. Examples referring to problems on the contact between plates and shells are presented in /1/, where there is also an extensive bibliography.

Let us consider the problem of the contact between two thin elastic plates with the bending stiffnesses a_1 and a_2. Let them occupy a domain Ω and be separated by a distance $\delta \geqslant 0$ in the natural (undeformed) state. For definiteness, let the plates also be rigidly clamped on the boundary $\partial\Omega$.

We define a closed convex set $K = \{u, v \in H_0^2(\Omega) \mid u - v \geqslant -\delta$ almost everywhere in $\Omega\}$ in $H_0^2(\Omega) \times H_0^2(\Omega)$ and we let $\Pi(u, v)$ denote a functional of the energy of the two plates

$$\Pi(u,v) = \langle a_1(\Delta u)^2 + a_2(\Delta v)^2 - 2Fu - 2Gv\rangle_\Omega, \quad \langle\cdot\rangle_\Omega = \int_\Omega (\cdot)\,dx$$

Here $H_0^2(\Omega)$ is the S.L. Sobolev space of functions having derivatives to second order inclusive that are summable in Ω and are zero on $\partial\Omega$ together with the first derivatives, u is the deflection of the upper plate, v of the lower, and F, G are external loads. We assume that the boundary $\partial\Omega$ is sufficiently smooth, and F, G belong to the space $L^2(\Omega)$.

The solution $(u, v) \in K$ of the problem of minimizing the functional Π in the set K exists and satisfies the variational inequality

$$\begin{gathered}\langle a_1\Delta u\,(\Delta u' - \Delta u) + a_2\Delta v\,(\Delta v' - \Delta v) - F\,(u' - u) - \\ G\,(v' - v)\rangle_\Omega \geqslant 0 \quad \forall\,(u',\ v') \in K\end{gathered} \tag{1}$$

This inequality is the necessary and sufficient condition for a minimum. It can also be obtained directly without relying on the variational formulation. The equilibrium equation for the upper plate has the form $a_1\Delta^2u - F = p$, where $p \geqslant 0$ is the pressure of the lower plate on the upper one. Analogously, for the lower plate $a_2\Delta^2v - G = -p$. Therefore, for arbitrary smooth functions u', v' satisfying the inequality $u' - v' \geqslant -\delta$ and equal to zero together with the first derivatives on $\partial\Omega$, the following relationship is valid

$$\begin{gathered}\langle(a_1\Delta^2u - F)(u' - u) + (a_2\Delta^2v - G)(v' - v)\rangle_\Omega = \\ \langle p\,(u' - v' - u + v)\rangle_\Omega\end{gathered}$$

The right side is non-negative. Indeed, if there is no contact at the point x_0, then $p(x_0) = 0$. If x_0 is a contact, then $u(x_0) - v(x_0) = -\delta$. And since $u'(x_0) - v'(x_0) \geqslant -\delta$, then $u'(x_0) - v'(x_0) - u(x_0) + v(x_0) \geqslant 0$.

The reverse assertion will be proved below. If the inequality (1) is satisfied, then the pressure of the upper on the lower plate equals the pressure of the lower plate on the upper one (the condition of agreement of corresponding measures).

Let us note that the presence of the variational formulation of the problem also permits utilization of modern optimization methods. An analogous approach is developed in /2/ for a linear elastic solid. In contrast to the classical Signorini problem on the contact between a linear elastic and a rigid body, the gap between elastic bodies in /2/ can be greater than zero.

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.140-146,1983

For $\varphi \geqslant 0, \varphi \in H_0^2(\Omega)$ and $\varepsilon > 0$ the pair of functions $(u + \varepsilon\varphi, v)$ belongs to K, and consequently, $\Pi(u + \varepsilon\varphi, v) \geqslant \Pi(u, v)$. We hence obtain

$$\langle a_1 \Delta u \Delta\varphi + (1/2) a_1 \varepsilon (\Delta\varphi)^2 - F\varphi \rangle_\Omega \geqslant 0$$

Passing to the limit as $\varepsilon \to 0$, we conclude that the quantity $\nu \equiv a_1 \Delta^2 u - F$ is a positive generalized function, and therefore, is a measure in the domain Ω /3/. This means that $\nu(B) < +\infty$ for an arbitrary compact $B \subset \Omega$. It is analogously proved that $-(a_2 \Delta^2 v - G)$ is also a measure. These measures agree. In fact $(u + \varepsilon\varphi, v + \varepsilon\varphi) \in K$, $\varphi \in H_0^2(\Omega)$, $\varepsilon > 0$, and therefore

$$\langle a_1 \Delta u \Delta\varphi - F\varphi + a_2 \Delta v \Delta\varphi - G\varphi \rangle_\Omega \geqslant 0$$

Because of the arbitrariness of φ, we hence obtain agreement between the mentioned measures.

The physical meaning of the measure ν is the following: $\nu(B)$ is the intensity of the action of one plate on the other in the set B.

We denote the contact set by $C = \{x \in \Omega \mid u(x) - v(x) = -\delta\}$. Correspondingly, $N = \Omega \setminus C$ is the non-contact domain.

The carrier $S(\nu)$ of the measure ν is lumped in the set C. This follows from the fact that the equilibrium conditions are satisfied in N (in the sense of distributions)

$$a_1 \Delta^2 u = F, \ a_2 \Delta^2 v = G$$

The following assertion is valid: if $F/a_1 - G/a_2 > 0$, then the contact set has no inner points.

In fact, we otherwise have $\Delta^2 u = \Delta^2 v$ in the neighborhood of the point of contact, and this means

$$\nu/a_1 - \nu/a_2 = \Delta^2 u - F/a_1 - \Delta^2 v + G/a_2 = G/a_2 - F/a_1 < 0$$

This contradicts the fact that ν is a measure.

The assertion proved means that when the required inequality is satisfied there is no circle of arbitrarily small radius all of whose points are contact points. It will hence follow in the axisymmetric case that if contact points generally exist, then they form a set of circles with radii $r_1 < r_2 < \ldots$, where there is not contact in the domain $r_i < r < r_{i+1}$.

It turns out that if the contact point is isolated, then the pressure of one plane on the other at this point equals zero. Namely, if $a_1 = a_2$, then the plates cannot have contact at an isolated point belonging to $S(\nu)$.

We use Lemma 4 (see below), according to which $\Delta u - \Delta v \in L^\infty_{\text{loc}}(\Omega)$ for the proof. Since $\Delta^2(u + v) = (F + G) a_1^{-1}$, then $\Delta u + \Delta v \in L^\infty_{\text{loc}}(\Omega)$, and therefore, $\Delta u \in L^\infty_{\text{loc}}(\Omega)$. Let x_0 be an isolated contact point. Let B_r denote a circle of radius r with center at x_0. We consider the equation $a_1 \Delta^2 u = F$ satisied in $B_r^0 = B_r \setminus \{x_0\}$. We prove that this equation is satisfied in B_r. It will hence follow that $x_0 \overline{\in} S(\nu)$. Let $F_0 \in H_0^2(\Omega) \cap H_0^1(\Omega)$ be a solution of the problem

$$\Delta F_0 = F, \ F_0 |_{\partial B_r} = 0$$

The equation $\Delta(a_1 \Delta u - F_0) = 0$ is valid in the domain B_r^0. From the results concerning the internal regularity for the biharmonic equation it follows that the function Δu is continuous in B_r^0. Because of the imbedding theorem the function F_0 is continuous in B_r^0. Since the quantity $a_1 \Delta u - F_0$ is bounded in B_r^0, then from the theorem on eliminable singularities for harmonic functions we obtain $\Delta(a_1 \Delta u - F_0) = 0$ in B_r.

The solution of contact problems is not generally smooth. In particular, the example /4/ (see also /5-6/) in the problem of equilibrium over an obstacle, where it is established that the solution (with a specific obstacle) does not belong to the space $W^3_{3,\text{loc}}(\Omega)$. In this case the following lemma is valid.

Lemma 1. The imbedding $u, v \in H^3_{\text{loc}}(\Omega)$ holds.

Proof. Let $\Omega_2 \subset \Omega_1 \subset \Omega$ be such domains that $\rho(\partial\Omega_1, \partial\Omega) \geqslant q > 0$, $q = \text{const}$. We select the function $\varphi \in C_0^\infty(\Omega_1)$, $\varphi \equiv 1$ on Ω_2, $\varphi \geqslant 0$, $|\varphi| \leqslant 1$ everywhere and we introduce the notation

$$\Delta_{i\tau} h(x) = [h(x + \tau e_i) - 2h(x) + h(x - \tau e_i)] \tau^{-2}$$

(e_i are the unit vectors). Let $0 < \lambda \leqslant (1/2)\tau^2$. We set

$$u_\lambda = u + \lambda\varphi^2 \Delta_{i\tau} u, \ v_\lambda = v + \lambda\varphi^2 \Delta_{i\tau} v$$

Taking account of the inequality $1 - 2\lambda/\tau^2 \geqslant 0$, we verify that $u_\lambda - v_\lambda \geqslant -\delta$, i.e., $(u_\lambda, v_\lambda) \in K$. We now substitute $(u', v') = (u_\lambda, v_\lambda)$ in (1). We obtain

$$\langle a_1 \Delta u \Delta(\varphi^2 \Delta_{i\tau} u) + a_2 \Delta v \Delta(\varphi^2 \Delta_{i\tau} v) \rangle_\Omega \geqslant \langle F\varphi^2 \Delta_{i\tau} u + G\varphi^2 \Delta_{i\tau} v \rangle_\Omega \tag{2}$$

We introduce the notation

$$d_{i\tau}h(x) = [h(x+\tau e_i) - h(x)]\,\tau^{-1}$$

The following chain is valid, where the difference between its successive terms either equals zero or has the following upper estimated quantity

$$c\,(\|u\|_2^2 + \|u\|_2\,\|d_{i\tau}(\varphi u)\|_2)$$

with constant c dependent only on the domain Ω and the function φ:

$$\langle \Delta u \Delta(\varphi^2 \Delta_{i\tau} u)\rangle_\Omega \to \langle \Delta(\varphi u)\,\Delta(\Delta_{i\tau}\varphi u)\rangle_\Omega \to \langle \Delta(\varphi u)\,\Delta_{i\tau}\Delta(\varphi u)\rangle_\Omega \to$$
$$-\langle \Delta(\varphi u)\, d_{-i\tau} d_{i\tau}\Delta(\varphi u)\rangle_\Omega \to -\langle d_{i\tau}(\Delta(\varphi u))\, d_{i\tau}(\Delta(\varphi u))\rangle_\Omega \to$$
$$-\langle \Delta(d_{i\tau}(\varphi u))\,\Delta(d_{i\tau}(\varphi u))\rangle_\Omega$$

Analogous manipulations are valid for the second term on the left side of inequality (2). The right side is estimated by the Cauchy inequality. Since $\|w\|_2 \leqslant c\|\Delta w\|_0$ for an arbitrary function $w \in H_0^2(\Omega)$ with a constant independent of w, we then consequently obtain

$$\|d_{i\tau}(\varphi u)\|_2^2 + \|d_{i\tau}(\varphi v)\|_2^2 \leqslant c\,\{\|F\|_0^2 + \|G\|_0^2 + \|u\|_2^2 +$$
$$\|v\|_2^2 + \|u\|_2\,\|d_{i\tau}(\varphi u)\|_2 + \|v\|_2\,\|d_{i\tau}(\varphi v)\|_2\}$$

The constant c is here independent of τ. Boundedness of its left side follows from this inequality, and this means $\varphi u, \varphi v \in H^3(\Omega)$, i.e., $u, v \in H^3_{\mathrm{loc}}(\Omega)$. Lemma 1 is proven.

Lemma 2. The function Δu (or Δv) is semicontinuous from above (below) in the domain Ω.

Proof. Let $\psi \in H^4(\Omega) \cap H_0^2(\Omega)$ be the solution of the problem

$$a_1\Delta^2\psi = F;\ \psi = \partial\psi/\partial n = 0 \text{ on } \partial\Omega$$

If $u_\varepsilon, \psi_\varepsilon$ are averages with infinitely differentiable kernel /7/, the functions u, ψ (continued outside Ω with the conservation of smoothness) and $f = u - \psi$, then because of the Green's formula we have

$$\Delta f_\varepsilon(x) = \frac{1}{2\pi r}\int_{\partial B_r(x)} \Delta f_\varepsilon(y)\,dy - \frac{1}{2\pi}\int_{B_r(x)} \Delta^2 f_\varepsilon(y)\ln r\,|x-y|^{-1}\,dy \tag{3}$$

Here $B_r(x)$ is a circle of radius r with center at the point x, and $\partial B_r(x)$ is the boundary of $B_r(x)$. Since $\Delta^2 u - \Delta^2\psi \geqslant 0$ is the distribution sense, the $\Delta^2 f_\varepsilon = \Delta^2 u_\varepsilon - \Delta^2\psi_\varepsilon \geqslant 0$. Because $\ln r|x-y|^{-1} \leqslant \ln r_1|x-y|^{-1}$ for $r_1 \geqslant r$, from (3) and an analogous equality for r_1, we obtain

$$\frac{1}{2\pi r}\int_{\partial B_r(x)} \Delta f_\varepsilon(y)\,dy \leqslant \frac{1}{2\pi r_1}\int_{\partial B_{r_1}(x)} \Delta f_\varepsilon(y)\,dy$$

We multiply this inequality by rr_1 and then integrate once with respect to r between zero and r, then the second time with respect to r_1 between r and r_1. We will consequently have

$$\frac{1}{\pi r^2}\int_{B_r(x)} \Delta f_\varepsilon(y)\,dy \leqslant \frac{1}{\pi r_1^2}\int_{B_{r_1}(x)} \Delta f_\varepsilon(y)\,dy$$

Passing to the limit here as $\varepsilon \to 0$, we conclude that this inequality is valid for Δf. Since Δf is a summable function, then for almost all $x \in \Omega$

$$b_r(x) \equiv \frac{1}{\pi r^2}\int_{B_r(x)} \Delta f(y)\,dy \to \Delta f(x), \qquad r \to 0 \tag{4}$$

However, $b_r(x)$ is a continuous non-increasing function such that it can be considered that Δf is a function semi-continuous from above. Analogous reasoning can be performed for $q = v - \xi$, where $\xi \in H^4(\Omega) \cap H_0^2(\Omega)$ is a solution of the problem

$$a_2\Delta^2\xi = G;\ \xi = \partial\xi/\partial n = 0 \ \text{ in } \ \partial\Omega$$

and it can be shown that Δg is a function semicontinuous from below. Since $\Delta\psi, \Delta\xi \in C(\overline{\Omega})$ by virtue of the theorem imbedding, then Lemma 2 is proved.

Now we note that Δf is a subharmonic function in Ω. Indeed, let α be harmonic function in $B_\rho(x)$, that equals Δf on $\partial B_\rho(x)$. Then $\Delta f \leqslant \alpha$ in $B_\rho(x)$. In fact

$$\Delta f|_{\partial B_\rho(x)} \in H^{1/2}(\partial B_\rho(x))$$

hence, the function α exists. The inequality mentioned follows from the Green's formula for b_r after the passage to the limit as $r \to 0$ and taking account of the relationship $\Delta b_r \geqslant 0$ in Ω. It is analogously proved that Δg is a superharmonic function in Ω.

Lemma 3. For $x_0 \in C$ the inequality $\Delta u(x_0) \geqslant \Delta v(x_0)$ is valid.

Proof. We denote $w = u - v$. According to the Green's formula, the equality

$$w(x_0) = I_1 - I_2, \quad I_1 = \frac{1}{2\pi r}\int\limits_{\partial B_r(x_0)} w(y)\,dy, \quad I_2 = \frac{1}{2\pi}\int\limits_{B_r(x_0)} \Delta w(y) \ln r\,|x_0 - y|^{-1}\,dy$$

is valid.

Since $w(y) \geqslant -\delta$ and $w(x_0) = -\delta$, it then follows that $I_2 \geqslant 0$. From this inequality we conclude that there exists a sequence of points $y_i \in B_r(x_0)$ such that $\Delta w(y_i) \geqslant 0$ $(y_i \to y_r)$. Evidently $\Delta w = \Delta u - \Delta v$ is a function semicontinuous from above. Hence, after passing to the limit as $i \to \infty$, we obtain $\Delta w(y_r) \geqslant 0$. Letting r go to zero and again using the semicontinuity of Δw from above, we conclude that $\Delta u(x_0) \geqslant \Delta v(x_0)$. Lemma 3 is proved.

Lemma 4. For $a_1 = a_2$ the embedding $\Delta f - \Delta g \in L^\infty_{\mathrm{loc}}(\Omega)$ is valid.

Proof. Let $\varepsilon(x)$ be a Dirac delta function. The convolution of two generalized functions will be denoted by an asterisk. Furthermore, let $\bar{\Omega}_0 \subset \Omega$ and let ν_0' be a narrowing of the measure $\nu' = \nu/a_1$ on Ω_0. We consider the potential

$$H * \nu_0'(x) = \int\limits_{\Omega_0} H(x-y)\,d\nu'(y), \quad H(x) = \frac{1}{2\pi}\ln|x|^{-1}$$

By virtue of the Fubini theorem this potential is a locally Lebesgue summable function, since ν_0' is a finite measure. We introduce the function

$$\gamma(x) = \Delta f(x) - \Delta g(x) + 2H * \nu_0'(x), \quad x \in \Omega_0 \tag{5}$$

and we prove it harmonic in Ω_0. We use the equality $\Delta H(x) = -\varepsilon(x)$ as well as the associative property of the convolution for two finite cofactors. We have the chain of equalities

$$\begin{aligned}\Delta\gamma = \Delta\varepsilon * \gamma = \Delta^2 f - \Delta^2 g + 2\Delta\varepsilon * (H * \nu_0') = 2\nu_0' + 2(\Delta\varepsilon * H) * \\ \nu_0' = 2\nu_0' + 2(\varepsilon * \Delta H) * \nu_0' = 2\nu_0' - 2\varepsilon * \nu_0' = 0\end{aligned} \tag{6}$$

The relationship (5) is satisfied almost everywhere in Ω_0. Since the functions $-H * \nu_0'(x)$ and $\Delta f(x) - \Delta g(x)$ are subharmonic in Ω, while the mean value over a sphere of radius r with center at a given point converges for a subharmonic function to the value of the function at this point as $r \to 0$, then the equality (5) is satisfied for all $x \in \Omega_0$.

We now use the theorem of boundedness of the potential /3/. If the potential of the measure ν has an upper bound on the carrier $S(\nu)$, then it has an upper bound in all space. According to an earlier proof, the inequality $\Delta u(x_0) \geqslant \Delta v(x_0)$ at each point of the carrier x_0, hence for all $x_0 \in \Omega_1 \cap S(\nu)$, $\bar{\Omega}_1 \subset \Omega_0$, we obtain from the representation of (5)

$$2\int\limits_{\Omega_0} H(x_0 - y)\,d\nu'(y) = \gamma(x_0) + \Delta g(x_0) - \Delta f(x_0) < +\infty$$

Therefore, the potential of the measure ν_1' (the narrowing of ν' on Ω_1) has an upper bound for all x. It also has a lower bound on Ω_1. Therefore, we conclude from (5) that the function $|\Delta f - \Delta g|$ is bounded in the domain Ω_2, $\bar{\Omega}_2 \subset \Omega_1$. Lemma 4 is proved.

Theorem. If $\delta > 0$, $a_1 = a_2$ and $F - G \leqslant 0$, then the noncontact domain will be connected.

Proof. Since u, v are zero on the boundary and continuous in $\bar{\Omega}$, then the non-contact domain contains the neighborhood of the boundary $\partial\Omega$. We assume that the assertion of the theorem is not true. Then there exists a connected component N_1 of the noncontact domain N whose points cannot possibly be connected by a curve lying in N, with the mentioned neighborhood of the boundary. Because of Lemma 3, the following relationship is satisfied on the boundary ∂N_1

$$\Delta u(x_0) \geqslant \Delta v(x_0) \tag{7}$$

Moreover, the equations

$$a_1\Delta^2 u = F, \quad a_2\Delta^2 v = G \tag{8}$$

are valid in the domain N_1.

The inequality

$$\Delta u(x) \geqslant \Delta v(x), \quad x \in N_1 \tag{9}$$

follows from (7) and (8).

Noting that the equality $u(x) - v(x) = -\delta$ is satisfied on the boundary ∂N_1, and using the maximum principle, we conclude that $u - v \leqslant -\delta$ in N_1. This relationship contradicts the definition of N_1.

Let us prove the inequality (9). Let the domains Ω_1, Ω_2, Ω_3 be such that $\bar{N}_1 \subset \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, $\bar{\Omega}_2 \subset \Omega_3$, $\bar{\Omega}_3 \subset \Omega$. A representation analogous to (5) exists in the domain Ω_3. If the potential $H * \nu_3'(x)$ is written in the form

$$H * \nu_3'(x) = -p(x) + \frac{1}{2\pi} \int\limits_{\Omega_3 \setminus \Omega_1} \ln|x-y|^{-1} d\nu'(y) - \frac{\nu'(\Omega_1)\ln m}{2\pi}$$

$$p(x) = -\frac{1}{2\pi} \int\limits_{\Omega_1} \ln m|x-y|^{-1} d\nu'(y), \quad m = \mathrm{const} > \operatorname{diam}\Omega_1$$

then the representation mentioned acquires the form

$$\Delta f(x) - \Delta g(x) = 2p(x) + \beta(x), \; x \in \Omega_3 \tag{10}$$

($\beta(x)$ is a continuous function for $x \in \Omega_1$). We hence conclude that

$$p_r(x) = -\frac{1}{2\pi} \int\limits_{\Omega_1 \setminus B_r(x)} \ln m|x-y|^{-1} d\nu'(y), \quad x \in \Omega_1$$

will converge to $p(x)$ from above as $r \to 0$. According to the Egorov theorem, for arbitrary $\varepsilon > 0$ there exists a closed subset $\Omega_\varepsilon \subset \Omega_1$ such that $\nu'(\Omega_1 \setminus \Omega_\varepsilon) < \varepsilon$ and p_r converges uniformly to p on Ω_ε.

Furthermore, we set

$$p_{r,\varepsilon}(x) = -\frac{1}{2\pi} \int\limits_{\Omega_1 \setminus B_r(x)} \ln m|x-y|^{-1} d\nu_\varepsilon'(y)$$

$$p_\varepsilon(x) = -\frac{1}{2\pi} \int\limits_{\Omega_1} \ln m|x-y|^{-1} d\nu_\varepsilon'(y)$$

where ν_ε' is the narrowing of the measure ν' on Ω_ε. Taking account of the uniform convergence of p_r to p on Ω_ε we obtain

$$0 \leqslant p_{r,\varepsilon}(x) - p_\varepsilon(x) \leqslant \frac{1}{2\pi} \int\limits_{\Omega_1 \cap B_r(x)} \ln m|x-y|^{-1} d\nu'(y) \to 0$$

uniformly on Ω_ε as $r \to 0$, so that the functions p_ε are continuous on Ω_ε. Let us note that $S(\nu_\varepsilon') \subset \Omega_\varepsilon$. Therefore, we conclude by the theorem on the continuity of potentials /3/ that the p_ε are continuous on Ω_1.

Since $p_\varepsilon \geqslant p$, then $2p_\varepsilon + \beta \geqslant \Delta f - \Delta g$ in Ω_1 by virtue of (10). In particular, this inequality is satisfied in ∂N_1. Then taking (7) into account, we obtain $2p_\varepsilon + \beta + \Delta\psi - \Delta\xi \geqslant 0$ on ∂N_1. Since $\Delta p_\varepsilon = 0$ and $\Delta\beta = 0$ in N_1, then $\Delta(2p_\varepsilon + \beta + \Delta\psi - \Delta\xi) = \Delta^2\psi - \Delta^2\xi = F/a_1 - G/a_2 \leqslant 0$ in N_1. By virtue of the continuity of $2p_\varepsilon + \beta + \Delta\psi - \Delta\xi$ in N_1, we obtain from the maximum principle

$$2p_\varepsilon + \beta + \Delta\psi - \Delta\xi \geqslant 0 \text{ in } N_1 \tag{11}$$

Since $S(\nu') \cap N_1 = \varnothing$, then for any $x \in N_1$

$$0 \leqslant p_\varepsilon(x) - p(x) \leqslant \frac{1}{2\pi} \int\limits_{\Omega_1} \ln m|x-y|^{-1} d(\nu' - \nu_\varepsilon') \to 0, \quad \varepsilon \to 0$$

Hence, $\Delta\psi(x) - \Delta\xi(x) \geqslant \lim_{\varepsilon\to 0}(-2p_\varepsilon(x) - \beta(x)) = -2p(x) - \beta(x)$, $x \in N_1$ follows from (11). Comparing (11) to (10) we have the inequality (9).

In conclusion, we consider the case of contact between a shallow shell and a plate. Let

$$\varepsilon_{11} = u_{x_1} + k_1 w, \; \varepsilon_{22} = v_{x_2} + k_2 w, \; \varepsilon_{12} = u_y + v_x$$

be the strain of the shell middle surface, and

$$N_{11} = \frac{Eh}{1-\sigma^2}(\varepsilon_{11} + \sigma\varepsilon_{22}), \qquad N_{22} = \frac{Eh}{1-\sigma^2}(\varepsilon_{22} + \sigma\varepsilon_{11})$$

$$N_{12} = \frac{Eh}{2(1+\sigma)}\varepsilon_{12}$$

the forces. Here u, v, w are shell displacements in the planes x_1, x_2 and the deflection, respectively, h is the thickness, E is the Young's modulus, σ is the Poisson's ratio and k_1, k_2 are the curvatures in the directions of the x_1, x_2 axes. Let W denote the plate deflection by considering, for definiteness, that the shell is above the undeformed plate in the undeformed state. Let the following support conditions also be satisfied on the boundary

$$u = v = w = \partial w/\partial n = W = \partial W/\partial n = 0$$

If $z = \Phi(x_1, x_2) \geqslant 0$ is the shape of the shell middle surface (Φ is a smooth function), then the solution u, v, w, W of the problem of minimizing the energy functional in the set of allowable displacements will satisfy the following relationships

$$\begin{aligned}&(w, W) \in K : \langle a_1 \Delta w (\Delta w' - \Delta w) + (k_1 N_{11} + k_2 N_{22} - \\ &\quad F)(w' - w) + a_2 \Delta W (\Delta W' - \Delta W) - G(W' - W)\rangle_\Omega \geqslant 0 \\ &\forall (w', W') \in K \\ &\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} = -f_1, \quad \frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} = -f_2\end{aligned} \tag{12}$$

where

$K = \{(w', W') \in H_0^2(\Omega) \times H_0^2(\Omega) \mid w' - W' \geqslant -\Phi$ almost everywhere in $\Omega\}$.

Here f_1, f_2, F are given external loads on the shell along the x_1, x_2, z axes, respectively, G is the load on the plate along the z-axis, and a_1, a_2 are cylindrical stiffness of the shell and plate.

Assertions analogous to those presented in the case of contact of two plates are valid for the problem (12). Namely, if $F_0/a_1 - G/a_2 > 0$, then the contact set has no interior points, $F_0 = F - k_1 N_{11} - k_2 N_{22}$. If $\Phi > 0$ on the boundary of the domain $\partial\Omega$, $a_1 = a_2$ and $F_0 - G \leqslant 0$, then the noncontact domain will be connected.

The proof of these assertions is executed exactly the same as the above by using the measures definable naturally in the domain Ω in this case.

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Translated by M.D.F.